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WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS OF HOPFIELD ARTIFICIAL NEURAL NETWORKS WITH LEAKAGE DELAY TERMS

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Abstract. We introduce high-order Hopfield neural networks with Leakage delays. Furthermore, we study the uniqueness and existence of Hopfield artificial neural networks having the weighted pseudo almost periodic forcing terms on finite delay. Our analysis is based on the differential inequality techniques and the Banach contraction mapping principle.

1. Introduction

The notion of almost periodic functions was introduced in the literature around 1925 the Denish mathematician Harald Bohr. A various types of almost periodic systems describe our world more realistically than periodic ones. This theory has undergone several interesting, natural, and powerful generalization, such as pseudo almost periodicity, weighted pseudo almost periodicity and so on.
In 1984, J. Hopfield [6] proposed that the dynamic behavior of neurons should be described with a set of ordinary differential equations or functional differential equations. Since then the Hopfield neural networks have been successfully applied to signal and image processing, pattern recognition and optimization. Hence they have attracted considerable attentions and many models for Hopfield neural networks introduced by many researchers. For more details on this field, we refer to ([2], [4], [11], [12], [15]) and references therein. And so their successful application requires an understanding of their long term behavior with dynamical properties, in specially, their existence, uniqueness and stability.

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In [2], C. Aouiti *et al.* investigate some sufficient conditions for the existence and uniqueness of weighted pseudo almost periodic solutions to the following systems for high order Hopfield neural networks given by

$$x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} d_{ij}(t) g_j(x_j(t)) + \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t-\tau))$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl}(t) g_j(x_j(t-\sigma)) g_l(x_j(t-\mu)) + I_i(t),$$

where n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at the time t.

Recently many authors investigated with the existence and exponential stability of positive almost periodic solutions of high-order Hopfield neural networks with time-varying delays as following

$$\begin{aligned} x' &= -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) g_j(x_j(t-\tau_{ij}(t))) \cdots\cdots\cdots (1) \\ &+ \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl}(t) g_j(x_j(t-\sigma_{ijl}(t))) g_l(x_j(t-\mu_{ijl}(t))) + I_i(t) \\ & \quad i = 1, 2, 3, \ldots, n. \end{aligned}$$

The neural system with variable coefficient rate $c_i(t)$ is a more general system than the above system. But some authors argue that the first term in each of the right side of (1) corresponds to stabilizing negative feedback of the system which acts instantaneously without time delay; these terms are variously known as "forgetting" or a leakage terms. The model which has time-varing leakage delays is more general than the previous ones(model). Therefore, some authors focused on the existence and stability of equibrium and periodic solutions for neural networks model involving leakage delays ([10], [12]). In recent years, C. Xu and P. Li [12] investigate the pseudo almost periodic solution of the following

Hopfield neural networks with time-varying leakage delays:

$$\begin{aligned} x' &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \cdots\cdots (2) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_j(t-\mu_{ijl}(t))) + I_i(t) \\ & i = 1,2,3,\ldots,n, \end{aligned}$$

The initial conditions associated with system (1) are of the form

$$x_i(s) = \varphi_i(s),\ s \in [-\tau, 0],\ i = 1,2,3,\cdots,n,$$

where $\varphi = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T \in C([-\tau, 0], \mathbb{R}^n)$, $\eta_i(t) \geq 0$ denotes the leakage delay, n is the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the ith unit at time t, $c_i(t)$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are first-order and second-order connection weights of the natural network, $\tau_{ij}(t) \geq 0$, $\sigma_{ijl}(t) \geq 0$ and $\nu_{ijl}(t) \geq 0$ correspond to the transmission delays, $I_i(t)$ denotes the external inputs at time t, and g_j is the activation function of signal transmission. The functions $c_i, a_{ij}, b_{ijl}, g_j, \tau_{ij}, \sigma_{ijl}, \nu_{ijl}, \eta_i$ are continuous on $\mathbb{R}$.

Motivated by the aforementioned works, we aim to establish some sufficient conditions to guarantee the existence and uniqueness of weighted pseudo almost periodic solutions for the system (2) having the pseudo almost periodic forcing terms.

This article is organized as follows: In the second section, we recall some definitions and results related with Weighted pseudo almost periodic functions. In third section, after providing some lemmas and preliminary results which will be used through this paper. We give a main results, inspired from the previous papers. In forth section, we give an example to illustrate our main results.

2. Preliminaries and notations

For a given $T > 0$ and each $\rho(weights) \in U$, set $\mu(T,\rho)= \int_{-T}^{T} \rho(t)dt$. In order to facilitate our discussion, we introduce the following notations:

$$\begin{aligned} \mathbb{U} &: \{\rho : \mathbb{R} \to (0,\infty) : \text{ locally integrable on } \mathbb{R} \text{ with } \rho > 0 \ (a.e.).\}, \\ \mathbb{U}_\infty &: \{\rho \in U : \lim_{T\to\infty} \mu(T,\rho) = \infty\}, \\ \mathbb{U}_b &: \{\rho \in U_\infty : \rho \text{ is bounded}, \inf_{x\in\mathbb{R}} \rho(x) > 0\}. \end{aligned}$$

Clearly, $\mathbb{U}_b \subset \mathbb{U}_\infty \subset \mathbb{U}$, with strict inclusions.
Note that $(BC(\mathbb{R},\mathbb{R}^n), ||\cdot||_\infty)$ is a Banach space where $||\cdot||_\infty$ denote the sup norm

$$||f||_\infty := \max_{1\le t\le n} \sup_{t\in\mathbb{R}} |f_i(t)|,$$

$BC(\mathbb{R},\mathbb{R}^n)$ denotes the space of bounded continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$. Conveniently, we introduce some notations. We will use $x = (x_1, \cdots, x_n)^T$ to denote a column vector, in which the symbol $(\cdot^T)$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, |x_2|, \cdots, |x_n|)^T$ and define $||x|| = \max_{1\le i\le n} |x_i|$. And we put $\varphi = \{\varphi_j(t)\} = (\varphi_1(t), \varphi_2(t), \cdots \varphi_n(t))^T$.
Next, we review some definitions and lemmas well known from our references ([1],[10], [12], [15]) and references therein.

Definition 2.1. A function $f \in BC(\mathbb{R},\mathbb{R})$ is called *almost periodic* if for every $\epsilon > 0$, if there exists an l such that every interval of length $l(\epsilon)$ contains a number τ with property that

$$||f(t+\tau) - f(t)|| < \epsilon, \text{ for every } t \in \mathbb{R}.$$

The collection of such functions is denoted by $AP(\mathbb{R},\mathbb{R})$.

Many authors have furthermore generalized the notion of almost periodicity in different directions.
The set of bounded continuous functions with vanishing mean value is denoted by $PAP_0(\mathbb{R})$, i.e., for $\rho \in \mathbb{U}_\infty$, define the weighted ergodic space

$$PAP_0(\mathbb{R}, X, \rho) = \{f \in BC(\mathbb{R},\mathbb{R}); \lim_{T\to\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)||f(t)||dt = 0\}$$

Definition 2.2. Let $\rho \in \mathbb{U}_\infty$. A continuous function $f \in BC(\mathbb{R},\mathbb{R})$ is *called weighted pseudo almost periodic* if it can be written as

$$f = h + \varphi,$$

with $h \in AP(\mathbb{R})$ and $\varphi \in PAP_0(\mathbb{R}, \mathbb{R}, \rho)$.
The collection of such functions is denoted by $WPAP_0(\mathbb{R}, \mathbb{R}, \rho)$.

DEFINITION 2.3. Let $x \in \mathbb{R}^n$ and $Q(t)$ be a $n \times n$ contineous matrix defined on $\mathbb{R}$. The linear system

$$x^{'}(t) = Q(t)x(t)$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants k, α and projection P and the fundamental solution matrix $X(t)$ of (1) satisfying

$$||X(t)PX^{-1}(s)|| \leq ke^{-\alpha(t-s)},$$
$$||X(t)(I-P)X^{-1}(s)|| \leq ke^{-\alpha(s-t)}.$$

LEMMA 2.4. [14] *Assume that* $Q(t)$ *is an almost periodic matrix function and* $g(t) \in PAP(\mathbb{R}, \mathbb{R}^n)$. *If the linear system* (2) *admits an exponential dichotomy, then pseudo almost periodic system*

$$x^{'}(t) = Q(t)x(t) + g(t)$$

has unique pseudo almost periodic solution $x(t)$, *and*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)ds + \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)ds.$$

LEMMA 2.5. [14] *Let* $c_i(t)$ *be an almost periodic function on* $\mathbb{R}$ *and*

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_{t}^{t+T} c_i(s)ds > 0, \; i = 1, 2, \cdots, n.$$

Then the linear system

$$x^{'}(t) = diag(-a_1(t), -a_2(t), \cdots, -a_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{R}$.

LEMMA 2.6. *If* $g_i(\cdot) \in PAP(\mathbb{R}, \mathbb{R})$ *satisfies the Lipschitz condition,* $x(\cdot) \in PAP(\mathbb{R}, \mathbb{R})$ *and* $\tau(\cdot) \in AP(\mathbb{R}, \mathbb{R})$ *then* $g_i(x(\cdot - \tau(\cdot))) \in PAP(\mathbb{R}, \mathbb{R})$.

LEMMA 2.7. *If* $\varphi(\cdot) \in PAP(\mathbb{R}, \mathbb{R}^n)$, *then* $\varphi(\cdot, -h) \in PAP(\mathbb{R}, \mathbb{R}^n)$.

LEMMA 2.8. *If* $\varphi, \psi \in PAP(\mathbb{R}, \mathbb{R})$, *then* $\varphi \times \psi \in PAP(\mathbb{R}, \mathbb{R})$.

Under the condition given by hypothesis, the decomposition is unique. Similar proofs and contents are well explained and can be easily proved.

THEOREM 2.9. *Let* $\rho_1, \rho_2 \in U_T$ *and* $\inf_{T>0} \frac{\mu(T,\rho_1)}{\mu(T,\rho_2)} > 0$. *Then the decomposition of weighted Stepanov pseudo almost periodic function is unique.*

3. Existence results for Stepanov weighted pseudo almost periodic mild solution

To work (1) we established some hypothesis and sufficient criteria as following:

(H_1) For $i, j, l = 1, 2, \cdots, n,\ c_i(t) : \mathbb{R} \to \mathbb{R}^+$ is a pseudo almost periodic function, $\eta_i(t), \tau_{ij}, \sigma_{ijl}, \mu_{ijl} : \mathbb{R} \to \mathbb{R}^+$ and $a_{ij}(t), b_{ijl}(t), I_i(t) : \mathbb{R} \to \mathbb{R}$ are weight pseudo almost periodic and there exists constants $I_i^*, a_{ij}^*, b_{ijl}^*, \eta_i^*$ such that

$$\sup_{t\in\mathbb{R}} |I_i(t)| = I_i^*, \sup_{t\in\mathbb{R}} |a_{ij}(t)| = a_{ij}^*, \sup_{t\in\mathbb{R}} |b_{ijl}(t)| = b_{ijl}^*, \sup_{t\in\mathbb{R}} |\eta_i(t)| = \eta_i^*.$$

(H_2) For all $j,\ 1 \le j \le n$, the functions g_j are weighted pseudo almost periodic function and there exist nonnegative constant L_j^g such that

$$||g_j(u) - g_j(v)|| \le L_j^g ||u - v||,\ |g_j(u)| \le M_j^g,\ g_j(0) = 0, \text{ for all } u, v \in \mathbb{R}.$$

(H_3) Let $\rho : \mathbb{R} \to \mathbb{R}^+, \rho \in U_\infty$ be continuous and

$$\sup_{T>0} \frac{\mu(T+\delta}{\mu(T,\rho)} < \infty \text{ and} \sup_{s\in\mathbb{R}} \frac{\rho(s)}{\rho(s+\delta)} < \infty \text{ for all } \delta \in \mathbb{R}.$$

(H_4) For all $i,\ 1 \le i \le n,,$

$$\sup_{T>0} \int_{-T}^{T} e^{c_i(T+t)} \rho(t) dt < \infty$$

(H_5) Assume that there exist nonnegative constants $p, q > 1$ and $k > 0$ such that

$$p = \sum_{j=1}^{n} a_{ij}^* g_j L_j^g + \sum_{l=1}^{n} b_{ijl}^* L_j^g M_l^g + \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl}^* L_j^g M_l^g < 1,$$

$$q = \max_{1\le i\le n} c_i^{-1} (c_i^* \eta_i^* + \sum_{j=1}^{n} a_{ij}^* L_j^g + \frac{2L}{1-\delta} \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl}^* L_j^g L_l^g) < 1$$

and

$$\max_{1\le i\le n} \{\frac{I_i^*}{c_i}\} = K,\ \frac{K}{1-\delta} \le 1.$$

We need the following results for proof of our main results.
From hypotheses and Theorem 5.11 in Zhang [14], implies that

$$\begin{cases} \varphi_{ij}(t-\tau_{ij}(t)) \in PAP(\mathbb{R},\mathbb{R}^n,\mu) \\ \varphi_{ij}(t-\tau_{ij}(t)) \in PAP(\mathbb{R},\mathbb{R}^n,\mu) \\ \varphi_{ij}(t-\tau_{ij}(t)) \in PAP(\mathbb{R},\mathbb{R}^n,\mu) \\ \eta_i(t)) \in PAP(\mathbb{R},\mathbb{R}^n,\mu). \end{cases}$$

Furthermore we obtain from Corollary 5.4 in Zhang [14] that

$$f(\varphi_{ij}(t-\tau_{ij}(t))) \in PAP(\mathbb{R},\mathbb{R}^n,\mu)$$

Also,

$$\begin{cases} c_i(s)\int_{s-\eta_i(s)}^{s}\varphi_i^{'}(u)du = c_i(t)\varphi_i(t) - c_i(t)\varphi_i(t-\eta_i(t)) \in PAP(\mathbb{R},\mathbb{R}^n,\mu). \\ \sum_{j=1}^{n} a_{ij}(t)g_j(\varphi_j(t-\tau_{ij}(t))) + I_i(t) \in PAP(\mathbb{R},\mathbb{R}^n,\mu). \\ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(\varphi_j(t-\sigma_{ijl}(t)))g_l(\varphi_j(t-\mu_{ijl}(t))) \in PAP(\mathbb{R},\mathbb{R}^n,\mu). \end{cases}$$

Next Lemma has already notioned by authors in Zhang [15], but we hereby provide detailed proof for completeness.

Lemma 3.1. *We assume that the hypothesis $(H_1)-(H_4)$ are satisfied. Define the nonlinear operator Γ as follows: for each $\varphi=(\varphi_1,\varphi_2,\cdots,\varphi_n)\in PAP(\mathbb{R},\mathbb{R}^n)$,$(\Gamma\varphi)(t) := x^{\varphi(t)}$, where*

$$x^{\varphi}(t) = \Big(\int_{-\infty}^{t} e^{-(t-s)c_1}F_1(s)ds, \int_{-\infty}^{t} e^{-(t-s)c_2}F_2(s)ds, \cdots, \int_{-\infty}^{t} e^{-(t-s)c_n}F_n(s)ds\Big)$$

and

$$\begin{aligned} F_i &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_j(t-\mu_{ijl}(t))) + I_i(t), \\ & \quad i = 1,2,3\ldots,n, \end{aligned}$$

Then Γ maps $PAP(\mathbb{R},\mathbb{R}^n,\rho)$ into itself.

Proof. Let $\varphi \in PAP(\mathbb{R},\mathbb{R}^n))$, by Hypothesis $(H_1)-(H_4)$, we have $\Gamma(\varphi)\in BC(\mathbb{R},\mathbb{R}^n)$ by according a similar method of Lemma 2.1 in [11]. Since $F_i \in PAP(\mathbb{R},\mathbb{R}^n)$ for $n = 1,2,\cdots,n$, let $F_i = F_{i1} + F_{i2}$ where

$F_{i1} \in AP(\mathbb{R}, \mathbb{R}^n)$ and $F_{i2} \in PAP_0(\mathbb{R}, \mathbb{R}^n)$

$$x_{\varphi_i}(t) = \int_{-\infty}^{t} e^{-(t-s)c_1} F_{i1}(s) + \int_{-\infty}^{t} e^{-(t-s)c_n} F_{i2}(s) = \Gamma_{i1}(t) + \Gamma_{i2}(t)$$

From the our hypothesis, $M[i] > 0$, by using Lemma $([2.4], [2.5])$, almost periodi system $x_i^{'}(t) = -c_i(t)x_i(t) + F_{i1}(t)$ has an almost periodic solution as following:

$$u(t) = \Big(\int_{-\infty}^{t} e^{-(t-s)c_1} F_{11}(s)ds, \int_{-\infty}^{t} e^{-(t-s)c_2} F_{21}(s)ds, \cdots, \int_{-\infty}^{t} e^{-(t-s)c_n} F_{n1}(s)ds\Big)^T$$

Therefore $T_{i1} \in AP(\mathbb{R}, \mathbb{R}^n)(i = 1, 2, 3 \cdots, n)$.
Next we show that $T_{i2} \in PAP_0(\mathbb{R}, \mathbb{R}^n)$, that is:

$$\lim_{T\to\infty} \frac{1}{\mu(-T,T)} \int_{-T}^{T} \Big| \int_{-\infty}^{t} e^{-(t-s)c_i} F_{i2} ds \Big| \rho(t)dt$$

By using (H_1),

$$\lim_{T\to\infty} \frac{1}{\mu(-T,T)} \int_{-T}^{T} \Big| \int_{-\infty}^{t} e^{-(t-s)c_i} F_{i2} ds \Big| \rho(t)dt \leq M(\rho) + N(\rho),$$

where

$$M(\rho) = \lim_{T\to\infty} \frac{1}{\mu(-T,T)} \int_{-T}^{T} \rho(t)\Big(\int_{-T}^{t} e^{-(t-s)c_i} |F_{i2}(s)|ds\Big)dt \text{ and}$$

$$N(\rho) = \lim_{T\to\infty} \frac{1}{\mu(-T,T)} \int_{-T}^{T} \rho(t)\Big(\int_{-\infty}^{-T} e^{-(t-s)c_i} |F_{i2}(s)|ds\Big)dt.$$

Next we obtain as following:

$$\begin{aligned} M(\rho) &= \lim_{T\to\infty} \frac{1}{\mu(-T,T)} \int_{-T}^{T} |F_{i2}|\rho(t)\Big(dt \int_{-T}^{t} e^{-(t-s)c_i} |F_{i2}|ds\Big)dt \\ &\leq \lim_{T\to\infty} \frac{1}{\mu(-T,T)} \int_{-T}^{T} |F_{i2}|\rho(t)\Big[\frac{1}{c_i}(1 - e^{c_i(t+T)})\Big]dt \\ &\leq \frac{1}{c_i} \lim_{T\to\infty} \frac{1}{\mu(-T,T)} \int_{-T}^{T} |F_{i2}|\rho(t)dt \\ &= 0, \end{aligned}$$

Similarly, by using Hypothesis (H_4),
we get:

$$\begin{aligned} N(\rho) &= \lim_{T\to\infty} \frac{M}{\mu(-T,T)} \int_{-\infty}^{-T} e^{c_i s}|F_{i2}|ds \int_{-T}^{-T} e^{c_i t}\rho(t)dt \\ &\leq \lim_{T\to\infty} \Big[\frac{M}{c_i\mu(-T,T)} \int_{-T}^{-T} e^{c_i t}\rho(t)dt \sup_{s\in\mathbb{R}} |F_{i2}|ds\Big] \\ &= 0. \end{aligned}$$

□

By using a similar argument in the proof of Theorem 3.1 in [11], By applying the similar mathematical analysis techniques in [11], we derive some new sufficient conditions ensuring the existence, uniqueness and of weighted pseudo almost periodic solutions of system (2).
Now it is time we give our main theorem.

THEOREM 3.2. *Assume that the hypotheses* $(H_1) - (H_2)$ *hold, then the equation* (2) *has a unique weighted pseudo periodic solution in the region* $\mathbb{E} = \{\varphi \in PAP(\mathbb{R}, \mathbb{R}^n, \rho), ||\varphi - \varphi_0||_\infty \leq \frac{\delta K}{1-\delta}\}$,
where

$$\varphi_0(t) = \Big(\int_{-\infty}^{t} e^{-(t-s)c_1} I_1(s)ds, \int_{-\infty}^{t} e^{-(t-s)c_2} I_2(s)ds, \cdots, \int_{-\infty}^{t} e^{-(t-s)c_n} I_n(s)ds\Big)^T.$$

Proof. For all $\varphi \in \mathbb{E}$, let $x^\varphi(t)$ be weighted pseudo almost periodic solution defined by

$$\begin{aligned} x_i^{'}(t) &= -c_i(t)\varphi_i(t) + c_i(t)\int_{t-\eta_i(t)}^{t} \varphi_i^{'}(s)ds + \sum_{j=1}^{n} a_{ij}(t)g_j(\varphi_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(\varphi_j(t-\sigma_{ijl}(t)))g_l(\varphi_j(t-\mu_{ijl}(t))) + I_i(t) \\ & \quad i = 1,2,3\ldots,n. \end{aligned}$$

From Lemma 2.4, Lemma 2.5 and hypothesis, we see that the equation (2) has a unique weighted pseudo almost periodic solution as following

$$\begin{aligned} x_i(t) &= \int_{-\infty}^{t} e^{-(t-s)c_i} \times \Big[c_i(s)\int_{s-\eta_i(s)}^{s} \varphi_i^{'}(u)du + \\ &+ \sum_{j=1}^{n} a_{ij}(t)g_j(\varphi_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(\varphi_j(t-\sigma_{ijl}(t)))g_l(\varphi_j(t-\mu_{ijl}(t))) + I_i(s)\Big]ds \end{aligned}$$

Now, we define a mapping T from $\mathbb{E}$ to $\mathbb{E}$ given by

$$T^{\varphi}(t) = x^{\varphi}(t) \text{ for any } \varphi \in \mathbb{E}$$

Since $\mathbb{E} = \{\varphi \in PAP(\mathbb{R}, \mathbb{R}^n, \rho), ||\varphi - \varphi_0||_\infty \leq \frac{1}{2}\}$ By using the definition of the norm of Banach space $PAP(\mathbb{R}, \mathbb{R}^n, \mu)$, we obtain

$$\begin{aligned} ||\varphi_0||_\infty &= \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}}\{||\int_{-\infty}^{t} e^{-(t-s)c_i} I_i(s)ds||\} \\ &\leq \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}} ||\{\frac{I_i^*}{c_i}||\} \\ &= \max_{1\leq i\leq n}\{\frac{I_i^*}{c_i}\} = K \end{aligned}$$

Then we get

$$\begin{aligned} ||\varphi||_\infty &= ||\varphi - \varphi_0||_\infty + ||\varphi_0||_\infty \\ &\leq \frac{\delta K}{1-\delta} + K \\ &\leq 1, \text{ for all } \varphi \in \mathbb{E}. \end{aligned}$$

Now, we the mapping T is a contraction mapping of $\mathbb{E}$. Firstly we show that $T^{\varphi} = x^{\varphi}$ for all $\varphi \in \mathbb{E}$.

We have that

$$\begin{aligned}
||T_\varphi - \varphi_0||_\infty &= \max_{1\le i\le n} \sup_{t\in\mathbb{R}} \Big|\Big| \int_{-\infty}^{t} e^{-(t-s)c_i} \times \Big[c_i(s)\int_{s-\eta_i(s)}^{s} \varphi_i^{'}(u)du + \\
&+ \sum_{j=1}^{n} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_j(t-\mu_{ijl}(t)))\Big]ds\Big|\Big| \\
&\le \max_{1\le i\le n} \sup_{t\in\mathbb{R}} \Big\{\Big\{ \int_{-\infty}^{t} e^{-(t-s)c_i} \times \Big[c_i^*\eta_i^*||\varphi||_\infty + \sum_{j=1}^{n} a_{ij}^* g_j L_j^g ||\varphi||_\infty \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* L_j^g M_l^g ||\varphi||_\infty^2\Big]ds \\
&\le \max_{1\le i\le n} c_i^{-1}\sum_{j=1}^{n} a_{ij}^* g_j L_j^g + \sum_{l=1}^{n} b_{ijl}^* L_j^g M_l^g \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}^* L_j^g M_l^g ||\varphi||_\infty \\
&\le p\cdot||\varphi||_\infty \le \frac{1-p}{pl},
\end{aligned}$$

which means that $T^\varphi = x^\varphi \in \mathbb{E}$.
Lastly, in view of hypotheses we show that T is a contraction mapping from $\mathbb{E}$ to $\mathbb{E}$. For any $\varphi, \psi \in \mathbb{E}$ and $i = 1, 2, \cdots, n$, we have

$$\begin{aligned}
& ||(T_\varphi - T_\psi)_i(t)||_\infty \\
&= \Big(||(T_\varphi - T_\psi)_1(t)||, \cdots, ||(T_\varphi - T_\psi)_n(t)||\Big)^T \\
&= \Big(\int_{-\infty}^{t} e^{-(t-s)c_1} \times \Big[c_1(s)\int_{s-\eta_1(s)}^{s} (\varphi_1^{'}(u) - \psi_1^{'}(u))du \\
&+ \sum_{j=1}^{n} a_{1j}(t)g_j(\varphi_j(s-\tau_{1j}(s))) - g_j(\psi_j(s-\tau_{1j}(t)))) \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{1jl}(t)g_j(x_j(t-\sigma_{1jl}(t)))g_l(x_j(t-\mu_{1jl}(t)))
\end{aligned}$$

$$
\begin{aligned}
- \quad & g_j(x_j(t-\sigma_{1jl}(t)))g_l(x_j(t-\mu_{1jl}(t)))\Big]ds\Big|\Big\}, \\
\cdots, \quad & \int_{-\infty}^{t} e^{-(t-s)c_n} \times \Big[c_n(s)\int_{s-\eta_n(s)}^{s}(\varphi_n'(u)-\psi_n'(u))du \\
+ \quad & \sum_{j=1}^{n} a_{nj}(t)g_j(\varphi_j(s-\tau_{nj}(s))) - g_j(\psi_j(s-\tau_{nj}(t)))) \\
+ \quad & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{njl}(t)g_j(x_j(t-\sigma_{njl}(t)))g_l(x_j(t-\mu_{njl}(t))) \\
- \quad & g_j(x_j(t-\sigma_{njl}(t)))g_l(x_j(t-\mu_{njl}(t)))\Big]ds\Big|\Big)^T \\
\leq \quad & \Big(\int_{-\infty}^{t} e^{-(t-s)c_1} \times \Big[c_1^*\eta_1^*||\varphi-\psi||_\infty + \sum_{j=1}^{n} a_{1j}^* L_j^g \sup_{t\in\mathbb{R}} ||\varphi-\psi||_\infty \\
+ \quad & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{1jl}^* L_j^g L_l^g \sup_{t\in\mathbb{R}} ||\varphi_l(t)||_\infty + (\sup_{t\in\mathbb{R}} ||\varphi_l(t)||_\infty)||\varphi-\psi||_\infty\Big]ds, \\
\cdots, \quad & \Big(\int_{-\infty}^{t} e^{-(t-s)c_n} \times \Big[c_n^*\eta_n^*||\varphi-\psi||_\infty + \sum_{j=1}^{n} a_{nj}^* L_j^g \sup_{t\in\mathbb{R}} ||\varphi-\psi||_\infty \\
+ \quad & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{njl}^* L_j^g L_l^g \sup_{t\in\mathbb{R}} ||\varphi_l(t)||_\infty + (\sup_{t\in\mathbb{R}} ||\varphi_l(t)||_\infty)||\varphi-\psi||_\infty\Big]ds\Big)^T \\
\leq \quad & \Big(c_1^{-1}(c_1^*\eta_1^* + \sum_{j=1}^{n} a_{1j}^* L_j^g + \frac{K}{1-\delta}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{1jl}^* L_j^g L_l^g)\cdot||\varphi-\psi||_\infty, \\
\cdots, \quad & c_1^{-1}(c_n^*\eta_n^* + \sum_{j=1}^{n} a_{1j}^* L_j^g + \frac{K}{1-\delta}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{njl}^* L_j^g L_l^g)\cdot||\varphi-\psi||_\infty)^T,
\end{aligned}
$$

Therefore

$$
\begin{aligned}
||(T_\varphi - T_\psi)_i(t)||_\infty \quad \leq \quad & \max_{1\leq i\leq n} c_1^{-1}(c_1^*\eta_1^* + \sum_{j=1}^{n} a_{1j}^* L_j^g \\
+ \quad & \frac{2L}{1-\delta}\sum_{j=1}^{n}\sum_{l=1}^{n} b_{1jl}^* L_j^g L_l^g)||\varphi-\psi||_\infty \\
= \quad & q\cdot||\varphi-\psi||_\infty.
\end{aligned}
$$

Since $q < 1$, it implies that $T : \mathbb{E} \to \mathbb{E}$ is a contraction mapping. By contraction mapping principle of $\mathbb{E}$, we obtain that the mapping T has a unique fixed point $z \in \mathbb{E}$ such that $Tz = z$, which means that the equation (1) has a unique weighted pseudo almost periodic solution. The proof of the theorem is completed. $\square$

4. Examples and An applications

In this section we consider a simple application of our abstracts results we give an modified example [10], [12] for $n = 2$ as the following Hopfield neural networks with time-varing leakage delays.

$$\begin{aligned} x_i'(t) &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{2} a_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{2}\sum_{l=1}^{2} b_{ijl}(t)g_j(x_j(t-\sigma_{ijl}(t)))g_l(x_j(t-\mu_{ijl}(t))) + I_i(t),\ i=1,2, \end{aligned}$$

where $c_1 = c_2 = 1, a_{11} = 0.1, a_{12} = 0.2, a_{21} = 0.3, a_{22} = 0.2, b_{111} = 0.2, b_{112} = 0.2, b_{121} = 0.2, b_{122} = 0.2, b_{211} = 0.2, b_{212} = 0.2, b_{221} = 0.2, b_{222} = 0.2, \tau_{ij} = sin^2 2t + \frac{0.01}{1+t^2}, \sigma_{ij}(t) = sin^2(5t) + \frac{0.03}{1+t^2}, \eta_{11} = sin^2 2t + \frac{0.01}{1+t^2}, \eta_{11} = sin^2 2t + \frac{0.01}{1+t^2}, \eta_{11} = sin^2 2t + \frac{0.01}{1+t^2}, \eta_{11} = sin^2 2t + \frac{0.01}{1+t^2}$, for all i,j,l=1,2.

We can show easily that all the conditions in our main theorem 3.2 are satisfied, which means the existence unique weighted pseudo almost periodic solution.

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